Journal of Sound and Vibration (2000) **230**(2), 468–475
doi:10.1006/jsvi.1999.2574, available online at http://www.idealibrary.com on IDEAL®

WORK/ENERGY-BASED STOCHASTIC EQUIVALENT LINEARIZATION WITH OPTIMIZED POWER

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(*Received* 8 *September* 1998 *and in final form* 16 *August* 1999)

1. INTRODUCTION

The stochastic equivalent linearization (SEL) method has been widely used in engineering applications since it was discovered by Booton [1] and Kazakov [2]. Numerous studies have been performed and documented in the context of this method, such as Caughey [3], Crandall [4], Wen [5], Spanos [6], Iwan *et al.* [7], and Kozin [8]. Extensive review of this subject may be found in the monographs by Lin [9], Roberts and Spanos [10] and in a review paper by Socha and Soong [11].

While the SEL method has been widely applied to solve various non-linear dynamic problems, its physics implications, numerical accuracy and efficient engineering applications still attract attentions in research and engineering communities. Therefore, many modified versions of SEL have been proposed (see, e.g., reference [12]). It may be mentioned, among others, that a work/energy-based SEL proposed by Zhang *et al.* [13] is not only better in computational accuracy for non-linear hardening-spring system but also consistent with the physics involved. For non-linear softening-spring system, the weighting factor/function or Monte Carlo simulation technique is introduced to various versions of SEL (e.g., reference [14–16]). Mathematically, the number of equivalent criteria is limitless, each of which might improve one statistical response or the other. In practice, however, not all of these criterions will be used *ad hoc*. Instead, only physics-based criterion can be used for engineering applications.

At this junction, let us look at the difference, in terms of the physics involved, between conventional and work/energy-based SEL. The former requires the minimization of mean square deviation of force between original non-linear and equivalent linear systems, while the latter requires the minimization of mean square deviation of energy and/or work between the two aforementioned systems. Apparently, equivalence in terms of force (restoring and/or damping force) does not imply the minimization of statistical responses between the two systems. The latter (not the former) is the primary purpose for the analysis of a stochastic dynamic system under random excitations. It is known that vibration is the process of exchange between its work and energy in a system, which is described by a different set of two parameters (e.g., mass and velocity, force and displacement). Apparently,

0022-460X/00/070468 + 08 $35.00/0

two-parameter-based work and energy of a system provide more comprehensive information about the dynamic system than the single-parameter-based force. Since the conventional equivalent criterion is based on the equation of motion (i.e., equivalence on one parameter such as the restoring or damping force term), which could be derived originally and also lose some information from the Lagrange's equation in the form of energy (kinetic and/or potential) and work (external and/or dissipated) of a dynamic system, it is expected and also to be proved later that the work/energy-based equivalence will generate better statistical dynamic responses than the conventional approach.

While the work/energy-based SEL method indeed improves the computational accuracy greatly in general, in comparison with the conventional method, there still exist some cases where it does not work very well. It is, therefore, the aim of this paper to present a more robust and unified version of work/energy-based SEL from the viewpoint of both physics and efficiency as well as effective engineering applications.

The new equivalent criterion requires the minimization of the mean square deviation of work and/or energy with power μ between original and equivalent systems. The parameter μ is so optimized that certain statistical responses of the original and equivalent dynamic systems subjected to simplified stochastic loads (usually white noise) will be the same. Apparently, the new equivalent criterion will not minimize the work and/or energy between the two systems subjected to general excitations (colored noise) either. Nevertheless, the associated difference will be expected to be small in comparison with other versions of SEL in most of its engineering applications, since the proposed criterion is exact at certain conditions (e.g., the white noise condition).

In order to find the power parameter μ, the exact solution of the FPK equations, for non-linear stochastic dynamic systems subjected to white noise excitations are needed. Fortunately, great progress on exact stationary response solutions for non-linear systems to white noise excitations has been made in the last decade (e.g., reference [17–25]). This lays a solid foundation for engineering applications of the proposed methodology. With the integration of exact solutions from pertinent FPK equations and work/energy-based SEL with parameter μ, the proposed version of SEL can be directly applied to solve almost any stochastic non-linear problems met in practical engineering.

2. EQUIVALENT CRITERION

Without loss of generality, consider the following governing equation for the non-linear vibration of a single-degree-of-freedom system:

$$\ddot{y} + \beta\dot{y} + \omega_1^2\, g(y) = f(t), \tag{1}$$

with ω_1 being the natural frequency of the system if $g(y) = y$.

The corresponding equivalent linear system is then controlled by

$$\ddot{y} + \beta\dot{y} + \omega_e^2\, y = f(t). \tag{2}$$

where ω_e is the equivalent natural frequency and can be found based on the proposed equivalent criterion

$$E[U^{\mu} - U_e^{\mu}]^2 = \text{min}. \tag{3}$$

In equation (3), E stands for ensemble average; U and U_e denote respectively the potentials of the original non-linear and equivalent linear systems per unit mass and can be found as

$$U(y) = \int_o^y \omega_1^2 g(u)\, \mathrm{d}u,\ U_e = \tfrac{1}{2}\, \omega_e^2 y^2. \tag{4,5}$$

The proposed equivalent criterion (3) indicates that the mean square deviation of the potential of power μ between original and equivalent systems is required to be minimized. This criterion is degenerated to the work/energy-based SEL [26] if $\mu = 1$ and to the conventional SEL if $\mu = 1$ and potentials U are replaced by the associated restoring forces.

We digress to comment on the physical implications of the proposed equivalent criterion as well as other pertinent ones. As known to all, the conventional criterion of SEL is based on the force equivalence in the governing equation of motion, which of course minimizes the force deviation between the original non-linear and equivalent linear systems. However, the main objective of SEL is to find, at the best estimation if no exact solution is available, the statistical responses of non-linear systems such as displacements and accelerations, not those of restoring forces. Apparently, minimization of restoring forces between original and equivalent systems does not imply the minimum of deviation of statistical responses in these two systems. From the viewpoint of the physics involved, particularly for work/energy of systems, vibration is the process of exchange between its work and energy in a system. The governing equation of motion, which is in the form of forces, can also be originally derived from Lagrange's equation in the form of energy and work of a system. In addition, the conventional equivalence is based on a sole parameter, i.e., the restoring force for the problem at hand, while the work/energy-based equivalence is based on two parameters, i.e., restoring force and displacement response. Therefore, the work/energy-based equivalent criterion thus sounds more reasonable and comprehensive than the force-based equivalent criterion. This is true in a general sense. However, like all the other approximate methods, there are no universal stochastic equivalent linearization methods applicable to all non-linear dynamic systems. Specifically, the detailed work/energy exchange pattern in a system with a different type of non-linear restoring force (e.g., hardening- or softening-type spring and strong or weak non-linearity) will be different. This implies that even the work/energy-based SEL will not result in the ubiquitous good estimation of statistical responses for all the non-linear cases.

To this end, the new equivalent criterion of equation (3) is proposed by using a power index μ, attempting to reflect inherent characteristics for different types of non-linear dynamic systems with different extent of non-linearity. Therefore, the objective of this study is to find the pertinent optimized values μ so that the statistical responses obtained through the proposed equivalent criterion will be as close to the true value as possible, within the prescribed framework.

Theoretically, it is almost impossible to know the optimization of parameter μ for the non-linear systems under colored stochastic excitations without using the Monte Carlo approach to obtain the corresponding statistical responses. Even if the Monte Carlo approach is applied, it might be unaffordable since huge computational effort will be needed to determine the optimized values of μ in each and every case that might be useful to engineering applications.

Instead, in this study, the parameter μ is obtained as described in the introduction and detailed as follows.

Specifically, equation (3) implies

$$\frac{\mathrm{d}}{\mathrm{d}\omega_e}\{E[U^\mu - U_e^\mu]^2\} = 0 \tag{6}$$

which yields, with the aid of equations (4) and (5),

$$\omega_e^2 = \left(\frac{E\,[(2U\,(y)\,y^2)^\mu]}{E\,[y^{4\mu}]}\right)^{1/\mu}. \tag{7}$$

It can be shown (e.g., reference [9]) that under the condition of white-noise Gaussian excitations with zero mean, the displacement variance $\sigma_{y(e)}^2$ can be found as

$$\sigma_{y\,(e)}^2 = \frac{\pi S_0}{\beta\omega_e^2}, \tag{8}$$

where S_0 is the constant two-sided power spectral density. Since the corresponding exact solution, denoted as $\sigma_{y(FPK)}^2$, can be obtained by solving the pertinent FPK equation, we could then find μ by letting

$$\sigma_{y\,(FPK)}^2 = \sigma_{y(e)}^2 \tag{9}$$

which leads to the following transcendental equation, with the use of equations (7) and (8):

$$\left(\frac{\sigma_{y\,(FPK)}^2}{\omega_1^2\,\sigma_1^2}\right)^\mu E[(2U(y)y^2)^\mu] - E\,[y^{4\mu}] = 0, \tag{10}$$

where

$$\sigma_1^2 = \frac{\pi S_0}{\beta\omega_1^2}. \tag{11}$$

For convenience in practical engineering applications, equation (8) could be further written in the form

$$\sigma_{y(e)}^2 = \eta_W\,\sigma_1^2, \tag{12}$$

where the factor η_W can be found from

$$\eta_W = \frac{\omega_1^2}{\omega_e^2}. \tag{13}$$

Equation (12) can be easily used to solve any kind of non-linear system for statistical responses to colored noise with a good approximation and to white noise with an exact solution, as long as the pertinent factor η_W is found in terms of the associated parameter μ obtained by equation (10).

3. EXAMPLES

Consider two typical non-linear systems: hardening- and softening-type non-linear restoring forces. The corresponding function $g(y)$ is

$$g(y) = (y + \varepsilon y^3) \tag{14}$$

$$g(y) = \begin{cases} y - \varepsilon y^3, & 0 \leqslant y < y_m, \\ \frac{2}{3}\ y_m, & y \geqslant y_m, \end{cases} \tag{15}$$

for a hardening-type system and

$$y_m = 1/\sqrt{3\varepsilon}. \tag{16}$$

for a softening-type system.

By solving equation (10) for these two types of non-linear systems, optimized values for the power parameter μ and consequently for factor η_W can be found, which are listed in Tables 1 and 2 for different $\varepsilon\sigma_1^2$.

Results from Table 1 indicate that the optimized values of μ will not be significantly changed with the value of $\varepsilon\sigma_1^2$, a comprehensive index for the extent of non-linearity, input as well as system characteristics. In addition, the optimized values μ will be roughly one for a hardening-type non-linear system and two for a softening-type non-linear system. This result verifies why the work/energy-based SEL works well for a variety of hardening-type non-linear systems and the square work/energy-related weighting factor/function-based SEL for softening-type non-linear systems.

TABLE 1

Optimized values of μ

$\varepsilon\sigma_1^2$	μ (hardening-type)	μ (softening-type)
0·05	1·156	1·550
0·1	1·110	1·748
0·5	1·006	2·102
1	0·969	2·190
5	0·918	2·266
10	0·904	2·277
50	0·888	2·285
100	0·883	2·286

TABLE 2

Optimized values of η_W

$\varepsilon\sigma_1^2$	η_W (hardening-type)	η_W (softening-type)
0·05	0·943	1·127
0·1	0·904	1·325
0·5	0·764	2·621
1	0·684	3·683
5	0·508	8·217
10	0·435	11·62
50	0·303	25·98
100	0·225	36·74

The factors η_W in Table 2 are obtained based on white noise input, which could provide, with the aid of equation (11), quick but yet good approximations for the calculation of statistical responses of a dynamic system under colored noise. This is particularly important and useful in reliability-based design and fatigue damage analysis of civil infrastructure systems under turbulent wind loads. This is partly because large numbers of structural components with non-linear properties will be involved in a practical design and any complicated time-consuming method for statistical response calculations should thus be avoided, and partly because the first vibration mode is dominant in the wind-induced response analysis and the proposed methodology could thus be easily applied. To increase the numerical accuracy for a dynamic system under colored noise, factors η_W can always be calculated based on equation (13) with the use of truly colored noise, instead of Table 2 with white noise, although the increased accuracy might be negligible for practical engineering applications.

4. CONCLUDING REMARKS

The proposed work/energy-based SEL with power μ as an adjustable parameter can be easily extended to the non-linear damping force term as well as multi-degree-of-freedom systems (e.g., reference [26]), which will be reported in future papers. An additional effort in computation of power parameter μ and factor η_W will be carried out for each and every non-linear case, where exact solutions of the pertinent FPK equation to white-noise excitations are available. Further research will also be performed on the effects of Gaussian and non-Gaussian colored noise on power parameter μ and factor η_W.

This papers purpose was to simplify analysis of practical engineering applications.

ACKNOWLEDGMENTS

This work was supported by the National Science Foundation with Grant Nos. CMS 9612127 and 9896070 with Dr S. C. Liu as a program director, the U.S.

Geological Survey with Award No. 98CRSA1077, and by two grants from the Colorado Advanced Software Institute that is sponsored by the Colorado Commission on Higher Education, an agency of the state of Colorado. The opinions, findings and conclusions expressed herein are those of the author and do not necessarily reflect the views of the sponsors.

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